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THE PROPAGATION OF SMALL PERTURBATIONS IN A VISCOELASTIC FLUID

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The propagation of small-amplitude waves is investigated in an incompressible viscoelastic fluid, the rheological behavior of which is described by a nonlinear differential-operator equation of state.

Waves in a linear viscoelastic medium have been discussed in detail in [1, 2]. In this paper, we consider the models of Oldroyd [3] and de Witt [4], and the generalizations of these models for the case of the finite spectrum of the times of relaxation and retardation. For the stated models an invariant formulation is adduced for the conditions of evolutionarity of a system of hydrodynamic equations. Possible types of short transverse waves are esteblished for media which possess transient elasticity. The phase polars and group polars of a point source are considered. The local characteristics are adduced for high-frequency transverse waves in the case of reflection and refraction at the boundary of an Oldroyd fluid with a linearly elastic solid.

Small perturbations are considered for the presence in the fluid of a stressed state which is different from the hydrostatic pressure.

1. Formulation of the conditions for evolutionarity of the hydrodynamic equations of models possessing a finite set of relaxation and retardation times. The system of dynamic equations for an incompressible viscoelastic fluid consists of the equation of continuity

$$\operatorname{div} \mathbf{v} = 0 \tag{1.1}$$

the equations of momenta

$$\rho\, d\mathbf{v}/dt = -\nabla p + \operatorname{div} \mathbf{T} + \rho \mathbf{F} \tag{1.2}$$

and the tensor equation of state

$$P_r\left(\frac{D}{Dt}\right)\mathbf{T} = 2\eta Q_s\left(\frac{D}{Dt}\right)\mathbf{E} \tag{1.3}$$

In Eq. (1.3), $\mathbf{T}$ is the tensor of "viscoelastic" stresses, $\mathbf{E}$ is the tensor of the rate of deformation; $P_r(D/Dt)$ and $Q_s(D/Dt)$ are differential operators representing polynomials of D/Dt

$$P_r\left(\frac{D}{Dt}\right) = \prod_{i=1}^{r}\left(1 + \lambda_i \frac{D}{Dt}\right), \quad Q_s\left(\frac{D}{Dt}\right) = \prod_{i=1}^{s}\left(1 + \theta_i \frac{D}{Dt}\right) \tag{1.4}$$

The quantities λ_i and θ_i form relaxation and retardation spectra, respectively. The symbol $D\mathbf{A}/Dt$ denotes the relative convective derivative of the tensor $\mathbf{A}$ defined by one of the following rules:

$$\frac{DA}{Dt} = \frac{dA}{dt} - (AE + EA) + \Omega A - A\Omega \qquad (1.5)$$

$$\frac{DA}{Dt} = \frac{dA}{dt} + (AE + EA) + \Omega A - A\Omega \qquad (1.6)$$

$$\frac{DA}{Dt} = \frac{dA}{dt} + \Omega A - A\Omega \qquad (1.7)$$

In the Cartesian system of coordinates, the components of the tensors $\mathbf{E}$ and $\mathbf{\Omega}$ have the form $\quad e_{ij} = {}^1/_2 (v_{i,j} + v_{j,i}), \quad \omega_{ij} = {}^1/_2 (v_{j,i} - v_{i,j})$

The symbol (1.5) denotes the "contravariant" derivative in Oldroyd formulation; the symbol (1.6) denotes the "covariant" derivative in Oldroyd formulation and the symbol (1.7) is the Jaumann-de Witt derivative.

Models of the type (1.3), in which the relative convective derivatives are replaced by partial derivatives with respect to time, are used in the theory of linear viscoelasticity. Similar models, containing nonlinear differential operators, are considered in [5, 6]. Using a simple mechanical model for the microstructure of a viscoelastic meterial, Blend [1] showed that there are two classes of linear variants of model (1.3). Media possessing transient elasticity for which $r = s + 1$ belong to the first calss. Models which exhibit viscous behavior when transiently stressed belong to the second class. For these models $r = s$. We shall consider similar classes for media with the equation of state (1.3) at the various laws of differentiation (1.5)–(1.7).

We shall consider the arbitrary continuous motion of a medium, the parameters of which $\mathbf{v}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ and $\mathbf{T}(\mathbf{x}, t)$ satisfy the system of equations (1.1), (1.3). We impose on the flow parameters small perturbations $\delta\mathbf{V}(\mathbf{x}, t)$, $\delta p(\mathbf{x}, t)$, $\delta\mathbf{T}(\mathbf{x}, t)$. With the usual assumptions made for linearization, we arrive at a system of linear equations with partial derivatives relative to the perturbations. The coefficients of this system depend on the parameters of the unperturbed flow.

We represent the perturbations in the vicinity of an arbitrary point of space of the independent variables $(\mathbf{x}_0, t_0)$ in the form of a plane wave

$$(\delta v_i, \delta p, \delta T_{ij}) = \mathrm{Re}\,[(w_i, q, \sigma_{ij}) \exp i(\mathbf{kx} - \omega t)] \qquad (1.8)$$

assuming that the complex amplitudes of the perturbations w_i, q and σ_{ij} are constant.

In order to obtain waves of the form of (1.8) as a solution, it is necessary to assume that the coefficients of the system of equations are constant. In this case, we assume the coefficients of the system in the fairly small vicinity of the point $(\mathbf{x}_0, t_0)$ to be equal to their values at this point. A similar procedure is justified when considering short-wave perturbations, for which at distances of the order of a wavelength the variation of the coefficients of the system is negligibly small. Then, in order to investigate evolutionarity of the system of equations, we can use the method of constant coefficients (see e. g. [7]).

When considering short waves and for the derivation of the conditions for evolutionarity in the linearized equations (1.1)–(1.3), it is sufficient to retain only terms with higher derivatives with respect to time and the coordinates of the perturbations δv_i, δp and δT_{ij}. The linearized equations of continuity and momenta then assume the form

$$\mathrm{div}\,\delta\mathbf{v} = 0, \qquad \rho\, d\delta\mathbf{v}/dt = -\nabla\delta p + \mathrm{div}\,\delta\mathbf{T} \qquad (1.9)$$

We should also add to Eqs. (1.9) the linearized equation of state. For this we shall consider two cases.

The first case is $r = s + 1$. The equations are being added for the rules of differentiation of (1.5)–(1.7) are of the form respectively

$$a\frac{d^{r-1}}{dt^{r-1}}\left(\frac{d\delta\mathbf{T}}{dt} - \delta\mathbf{E}\cdot\mathbf{T} - \mathbf{T}\delta\mathbf{E} + \delta\Omega\cdot\mathbf{T} - \mathbf{T}\delta\Omega\right) = 2\eta b\frac{d^{r-1}}{dt^{r-1}}\delta\mathbf{E}$$

$$a\frac{d^{r-1}}{dt^{r-1}}\left(\frac{d\delta\mathbf{T}}{dt} + \delta\mathbf{E}\cdot\mathbf{T} + \mathbf{T}\delta\mathbf{E} + \delta\Omega\cdot\mathbf{T} - \mathbf{T}\delta\Omega\right) = 2\eta b\frac{d^{r-1}}{dt^{r-1}}\delta\mathbf{E} \qquad (1.10)$$

$$a\frac{d^{r-1}}{dt^{r-1}}\left(\frac{d\delta\mathbf{T}}{dt} + \delta\Omega\cdot\mathbf{T} - \mathbf{T}\delta\Omega\right) = 2\eta b\frac{d^{r-1}}{dt^{r-1}}\delta\mathbf{E}$$

$$a = \prod_{i=1}^{r}\lambda_i, \quad b = \prod_{i=1}^{r-1}\theta_i\,(r > 1); \; a = \lambda, \; b = 1\,(r = 1)$$

By substituting solutions of the form (1.8) into Eqs. (1.9), we obtain

$$w_i n_i = 0, \qquad -\rho c w_i = -q n_i + \sigma_{ij} n_j \quad (c = \omega/k - \mathbf{vn}, \; \mathbf{n} = \mathbf{k}/k) \qquad (1.11)$$

By substituting solutions of the form (1.8) into Eqs. (1.10), we obtain

$$c^{r-1}\left[c\sigma_{ij} + \left(T_{kj}n_k + \eta\frac{b}{a}n_j\right)w_i + \left(T_{ik}n_k + \eta\frac{b}{a}n_i\right)w_j\right] = 0$$

$$c^{r-1}\left[c\sigma_{ij} + \eta\frac{b}{a}(n_j w_i + n_i w_j) - (T_{kj}n_i + T_{ik}n_j)w_k\right] = 0 \qquad (1.12)$$

$$c^{r-1}\left[c\sigma_{ij} + \left(\eta\frac{b}{a}n_j + \frac{1}{2}T_{kj}n_k\right)w_i + \left(\eta\frac{b}{a}n_i + \frac{1}{2}T_{ik}n_k\right)w_j - \right.$$
$$\left. - \frac{1}{2}(T_{kj}n_i + T_{ik}n_j)w_k\right] = 0$$

Each of the three groups of equations in (1.12) together with Eqs. (1.11), forms a system of ten linear algebraic homogeneous equations relative to ten perturbation amplitudes of w_i, q and σ_{ij}. The condition for the existence of nontrivial solutions for these systems is that the determinants of the stated systems vanish.

In order to evaluate the determinants, it is convenient to consider all quantities in a special rectangular Cartesian system of coordinates S. The axis x_1 of the system S is orientated along the vector $\mathbf{n}$. We select the directions of the axes x_2 and x_3 such that the nondiagonal element T_{23} of the unperturbed "viscoelastic" stress tensor is equal to zero. If the area x_2x_3 is not subjected to a uniform tension or compression, then a unique system S exists in which $T_{23} = 0$ and $T_{22} > T_{33}$. We relate all vector and tensor quantities to this system.

For the determinants Δ_1, Δ_2 and Δ_3 of the system of equations corresponding to the rules of differentiation of (1.5)–(1.7), we obtain the following expressions:

$$\Delta_1 = c^{6r-2}(-\rho c^2 + T_{11} + \eta b/a)^2$$

$$\Delta_2 = c^{6r-2}(\rho c^2 + T_{22} - \eta b/a)(\rho c^2 + T_{33} - \eta b/a)$$

$$\Delta_3 = c^{6r-2}(\rho c^2 - {}^1/_2(T_{11} - T_{22}) - \eta b/a)(\rho c^2 - {}^1/_2(T_{11} - T_{33}) - \eta b/a)$$

The nontrivial value of c^2 for a model with "contravariant" derivative in Oldroyd formulation is
$$c^2 = \rho^{-1}(T_{11} + \eta b/a)$$

The nontrivial values of c^2 in the model with "covariant" derivative in Oldroyd formulation are

$$c_+^2 = \rho^{-1}(\eta b/a - T_{33}), \qquad c_-^2 = \rho^{-1}(\eta b/a - T_{22})$$

For nontrivial values of c^2 in the model with the Jaumann derivative we obtain the formulas

$$c_+^2 = \rho^{-1}[{}^1/_2(T_{11} - T_{33}) + \eta b/a], \qquad c_-^2 = \rho^{-1}[{}^1/_2(T_{11} - T_{22}) + \eta b/a]$$

In our further considerations we shall call the quantity c–velocity of sound and we shall define the dependence of the quantity c on the direction $\mathbf{n}$ by the term "anisotropy of sound". Waves whose fronts are propagated in a given direction with velocities c_+ and c_- shall be called (by analogy, for example, with magnetohydrodynamics) fast and slow waves, respectively.

The system of equations is evolutionary if the condition $\operatorname{Im}\,\omega < \text{const}$ for $k \to \infty$ is satisfied for solutions of the form (1.8).

It is obvious that in our case the condition for evolutionarity is equivalent to the requirement that $c^2 > 0$ for each of the models being considered (*). Therefore, the conditions for evolutionarity for models with the derivatives (1.5), (1.6) and (1.7), respectively, have the form

$$T_{11} + \eta b/a > 0, \qquad -T_{22} + \eta b/a > 0, \qquad {}^1/_2\,(T_{11} - T_{22}) + \eta b/a > 0$$

As the direction of the vector $\mathbf{n}$ can be arbitrary, and the system S rotates together with the vector $\mathbf{n}$, the invariant form must be assigned to the inequalities written above. Let us assume that $T_1 \geqslant T_2 \geqslant T_3$ are the principal values of the tensor $\mathbf{T}$ at the point being investigated. Using the fact that the principal values of the tensor $\mathbf{T}$ realize extrema of the quadratic form $T_{ij}n_in_j$ defined on a unit sphere, we obtain for the model (1.3) with the derivative (1.5) the condition for evolutionarity in the form

$$T_3 > -\eta b/a \tag{1.13}$$

In deriving the invariant formulation for the condition of evolutionarity for a model with the derivative (1.6), we note that for a fixed direction of $\mathbf{n}$ the quantity T_{22} realizes a maximum of the quadratic form in a set of vectors formed by the intersection of the unit sphere with the plane x_2x_3. Therefore, a direction $\mathbf{n}$ exists for which $T_{22} = T_1$.

Consequently, the equations for the "covariant" model are evolutionary under the condition

$$T_1 < \eta b/a \tag{1.14}$$

The criterion for evolutionarity is established similarly for the model with Jaumann derivative

$${}^1/_2(T_1 - T_3) < \eta b/a \tag{1.15}$$

For two-constant models of the medium ($r = 1$, $s = 0$) the quantity $\eta b/a$ is equal to the shear modulus of the fluid $\mu = \eta/\lambda$, where λ is the unique stress relaxation time. The conditions that the equations of the two-constant model be evolutionary, interpreted

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*) The condition $\operatorname{Im}\,\omega / k \to 0$ for $k \to \infty$ at first sight seems to be inadequate for root extraction, as this condition is also fulfilled in the case when $\operatorname{Im}\,\omega \sim Ak^N$, where $A > 0$, $0 < N < 1$. However, such behavior of $\operatorname{Im}\,\omega$ is found to be not possible in consequence of the specific structure of the total variance equations, corresponding to linearized systems with the conditions of conservation in the latter, terms with the lowest derivatives.

as the conditions for a hyperbolic system of equations of uniform unsteady flow, were given in [8].

It is easy to see from inequalities (1.13)–(1.15) that the restrictions on the state of stress of the medium imposed by the requirements of evolutionarity are completely different for different models. In models containing differential Oldroyd operators, small perturbations start to increase with infinite rate if the compressing normal "viscoelastic" stresses exceed the value of $\eta b / a$ in the case of the "contravariant" model, or if the tensile normal stresses exceed this same value in the "covariant" model. In models with the differential Jaumann operator, a similar increase of small perturbations occurs at fairly high tangential stresses.

Second case, $r = s$. In this case, the system being examined consists of the equations of (1.9) and the linearized equation of state which, for any of the differential rules (1.5)–(1.7), has the form

$$a \frac{d^r}{dt^r} \delta \mathrm{T} = 2\eta b^* \frac{d^r}{dt^r} \delta \mathrm{E}, \qquad a = \prod_{i=1}^{r} \lambda_i, \quad b^* = \prod_{i=1}^{r} \theta_i \qquad (r \geqslant 1) \tag{1.16}$$

$$a = b^* = 1 \qquad (r = 0)$$

The determinant of the homogeneous system of equations which is obtained by substituting (1.8) into Eqs. (1.9), (1.16) is

$$\Delta = c^{6r} (\rho c + ik\eta b^* / a)^2$$

Therefore, the unique nontrivial quantity c is equal to $- ik\eta b^*/a\rho$. This value satisfies the condition of evolutionarity, as $\operatorname{Im} \omega < 0$. Consequently, the hydrodynamic equations of an incompressible medium (1.3), in the case of $r = s$, are evolutionary. In the case $r = s = 0$, we arrive at the well-known conclusion concerning the evolutionarity of the Navier-Stokes system of equations.

For the models (1.3) with $r = s$, the planes $t = \mathrm{const}$ are the characteristics of the system of equations. Therefore, perturbations of the parameters in such models can propagate with infinite velocity.

The conditions of evolutionarity were established above from analysis of the asymptotic variance equations, corresponding to short-wave perturbations. For models of the class $r = s + 1$, we can consider in place of the asymptotic variance equations, the characteristic equations of systems which describe uniform unsteady flow. Here the evolutionarity condition is equivalent to the requirement that the characteristic equation corresponding to the model being investigated has real roots. For models of the class $r = s$, we can not restrict our consideration to only one characteristic equation when examining the system of equations for evolutionarity.

We note that in order to determine the propagation velocity of waves of finite length in models of both classes, total variance equations must be considered.

2. Application of an "evolutionary" model in the problem of development of one-dimensional perturbations in a plane channel. Solutions are possible in the case $r = s + 1$, in which the conditions for evolutionarity are violated. Appropriate examples are given in [9, 10] for a two-constant Oldroyd "contraviariant" model. In this case, the equations of the original model, in general, cannot be used for describing the development of small perturbations, as the upper limit of the rate of increase of sinusoidal initial perturbations for values of k from the interval $(0, \infty)$ is found to be infinity.

In order to describe the development of small perturbations in the region where evolutionarity of the initial equations is not possible, the effect of supplementary physical parameters should be taken into account. In a real system, these parameters can be extremely small, but they may play a definite role in establishing a finite upper limit for the rate of buildup of perturbations.

The nonsteady-state problem concerning one-dimensional flow in a plane channel $0 < z < h$ is considered in [9] for a two-constant Oldroyd "contravariant" model

$$\mathbf{T} + \lambda D\mathbf{T}/Dt = 2\eta \mathbf{E} \tag{2.1}$$

The problem is reduced to finding the solution of a linear second-order equation relative to the longitudinal velocity u. This equation in the absence of a longitudinal pressure gradient has the form

$$\frac{\partial^2 u}{\partial t^2} - \frac{\tau_0 e^{-t/\lambda} + \eta/\lambda}{\rho} \frac{\partial^2 u}{\partial z^2} + \frac{1}{\lambda}\frac{\partial u}{\partial t} = 0, \qquad \tau_0 = \text{const} \tag{2.2}$$

The initial and boundary conditions for Eq. (2.2) are formulated in the form

$$u(z, 0) = u_0(z), \qquad (\partial u/\partial t)_{t=0} = u_1(z), \qquad u(0, t) = u(h, t) = 0 \tag{2.3}$$

The solution of this problem is constructed in the form of an infinite series

$$u(z, t) = F_1(t)\sin\alpha_1 z + F_2(t)\sin\alpha_2 z + \ldots, \qquad \alpha_k = k\pi/h \tag{2.4}$$

In the case when Eq. (2.2) at the instant $t = 0$ belongs to the elliptical type, the series (2.4) can prove to be divergent for any value of $t > 0$ of the ellipticity interval of Eq. (2.2). Consequently, model (2.1) becomes unsuitable for describing the process of development of small perturbations.

It should be noted that with a fairly "good" behavior of the functions $u_0(z)$ and $u_1(z)$ their Fourier coefficients for $k \to \infty$ can decrease so strongly that the series (2.4) will be convergent. In this case, the starting system of equations can supply certain data about the development of the initial perturbations. A similar approach is used in [11], where the problem with initial data for an elliptical equation is considered. In this case the rate of increase of the analytic initial distribution, decreasing at infinity, is limited down to a certain "critical" instant.

The problem of flow in a channel as formulated in [9] can be considered also for a three-constant "contravariant" Oldroyd model

$$\mathbf{T} + \lambda \frac{D\mathbf{T}}{Dt} = 2\eta\left(\mathbf{E} + \theta \frac{D\mathbf{E}}{Dt}\right) \tag{2.5}$$

It was established in Sect. 1 that the hydrodynamic equations for model (2.5) are evolutionary. In connection with this, it is interesting to examine the stated problem for model (2.5) in the region of ellipticity of Eq. (2.2) and to investigate the behavior of the perturbations for $\theta \to 0$. In the three-constant model, for the longitudinal velocity u we obtain the equation

$$\frac{\theta\eta}{\lambda\rho}\frac{\partial^3 u}{\partial t \partial z^2} - \left(\frac{\partial^2 u}{\partial t^2} - \frac{\tau_0 e^{-t/\lambda} + \eta/\lambda}{\rho}\frac{\partial^2 u}{\partial z^2} + \frac{1}{\lambda}\frac{\partial u}{\partial t}\right) = 0 \tag{2.6}$$

We preserve the initial and boundary conditions for Eq. (2.6) in the previous form (2.3). We also present the solution of problems (2.3) and (2.6) in the form of the series (2.4); however, the behavior of the function $F_k(t)$ will be significantly different from the behavior of the similar functions in the two-constant model. Without deriving precise formulas for $F_k(t)$, we shall note briefly the principal features in the behavior of these functions for models (2.1) and (2.5).

If the initial perturbations u_0 and u_1 have the form of simple harmonics $\sin \alpha_k z$, the functions $F_k(t)$ for the two-constant model in the ellipticity interval $(0, t_*)$ of Eq.(2.2) at $k \to \infty$ behave as $\exp(\alpha_k f(t))$. The function $f(t)$ is a positive monotonically increasing function in the interval $(0, t_*)$ and $f(0) = 0$. Thus, the rate of increase of the sinusoidal perturbations increases unlimitedly when the wavelength tends to zero.

In the three-constant model for the same simple initial perturbations and fixed value θ the functions $F_k(t)$ decrease in their absolute value in the interval $(0, t_*)$ if k is fairly large. Therefore, ultra-short wave perturbations are not responsible for the principal mechanism of oscillation buildup. In the three-constant model, the carriers of this mechanism prove to be "medium wave" perturbations. The wavelengths of these perturbations lie in an interval limits of which are dependent on θ, so that for $\theta > 0$ the lower limit of the interval is strictly positive. There is a "critical" wavelength within this interval, which also depends on θ and for which the corresponding function $F_k(t) \equiv F(t, \theta)$ has the maximum rate of increase. For $\theta \to 0$, the "critical" wavelength tends to zero causing the most "dangerous" short-wave perturbations. The function $F(t, \theta)$ for $\theta \to 0$ in the interval $(0, t_*)$ behaves as $\exp[\theta^{-1} \lambda g(t)]$, where $g(t)$ is a positive, monotonically increasing function in the interval $(0, t_*)$ and $g(0) = 0$. Consequently, in the three-constant model for the rate of increase of sinusoidal perturbations the finite upper limit exists. For $\theta \to 0$, this limit removes to infinity.

For $t \to \infty$, the perturbations in the two-constant and three-constant models damp. In model (2.1) the damping of the short-wave perturbations is of an oscillatory nature, and the logarithmic decrement of the oscillations tends to zero for $k \to \infty$. In model (2.5), waves of "medium" range have the character of damping oscillations for $t \to \infty$, and for the "critical" wavelength the decrement tends to a constant value for $\theta \to 0$. Damping of the ultra-short waves in model (2.5) is aperiodic.

An example of the application of the three-constant model (2.5) with subsequent limiting transition $\theta \to 0$ in the region of evolutionarity of model (2.1) is given in [10].

3. Types of short waves in media which possess transient elasticity. Phase polars and group polars. We shall consider media which correspond to the equation of state (1.3) for the case $r = s + 1$. For such media which possess finite propagation velocities of short wave perturbations we shall establish the possible types of plane short waves.

Considering all quantities in the S system of coordinates, introduced in Sect. 1, we find the general solution of the homogeneous systems of linear equations formed by Eqs. (1.11) and one of the groups of Eqs. (1.12). For the "contravariant" model the general solution for the perturbation amplitudes has the form

$$w_1 = 0, \quad q = 0, \quad \sigma_{11} = 0, \quad \sigma_{12} = -\rho c w_2, \quad \sigma_{13} = -\rho c w_3$$
$$\sigma_{22} = -2T_{12} w_2 / c, \quad \sigma_{23} = -(T_{13} w_2 + T_{12} w_3)/c, \quad \sigma_{33} = -2T_{13} w_3 / c \quad (3.1)$$
$$c = \pm [\rho^{-1} (T_{11} + \eta b/a)]^{1/2}$$

The complex amplitudes $w_2 = \alpha_2 + i\beta_2$ and $w_3 = \alpha_3 + i\beta_3$ in the formulas (3.1), which correspond to the velocity components in the transverse wave are arbitrary, all remaining values are linear combinations of these amplitudes.

The relations between the parameters of short transverse wave of small amplitude can be obtained from the relations of the tangential discontinuity, if we assume the intensity of the latter to be infinitely small. For example, in the case of the "contravariant" model (2.1), formulas (3.1) are obtained from the relations at the discontinuity in one-dimensional flow established in [8]. In the discontinuity relations it is necessary to neglect quadratic terms in relation to the magnitudes of the jumps and to substitute $\{v_i\}$ by w_i and $\{T_{ij}\}$ by σ_{ij}.

Relationships (3.1) show that in a transverse wave (1.8), the velocity vector $\delta\mathbf{v}$ is elliptically polarized. The parameters of the ellipse circumscribed by the extremity of the vector $\delta\mathbf{v}$ and its orientation depend on the complex amplitudes w_2 and w_3. If the condition $\alpha_2\alpha_3 + \beta_2\beta_3 = 0$ is satisfied, the axes of the ellipse coincide with the coordinate axes x_2, x_3. Linear polarization of the vector $\delta\mathbf{v}$ is obtained in the particular case when $\alpha_2\beta_3 - \alpha_3\beta_2 = 0$.

In the case of the "covariant" model the general solution for the perturbation amplitudes in fast waves has the form

$$\begin{aligned} &w_1 = w_2 = 0, && q = 2T_{13}w_3/c_+ \\ &\sigma_{11} = 2T_{13}w_3/c_+, && \sigma_{13} = -\rho c_+ w_3 \\ &\sigma_{12} = \sigma_{22} = \sigma_{23} = \sigma_{33} = 0, && c_+ = \pm\rho^{-1/2}[\eta b/a - T_{33}]^{1/2} \end{aligned} \tag{3.2}$$

and the general solution of the perturbation amplitudes in slow waves is presented in the form

$$\begin{aligned} &w_1 = w_3 = 0, && q = 2T_{12}w_2/c_- \\ &\sigma_{11} = 2T_{12}w_2/c_-, && \sigma_{12} = -\rho c_- w_2 \\ &\sigma_{13} = \sigma_{22} = \sigma_{23} = \sigma_{33} = 0, && c_- = \pm\rho^{-1/2}[\eta b/a - T_{22}]^{1/2} \end{aligned} \tag{3.3}$$

It follows from relations (3.2) and (3.3) that in the corresponding waves, the vector $\delta\mathbf{v}$ is linearly polarized. The plane of the oscillations of this vector cannot be arbitrary as in the case of the "contravariant" model. In a fast wave this is the coordinate plane x_1x_3, and in the slow wave it is the coordinate plane x_1x_2. In the "covariant" model composition of two mutually perpendicular oscillations does not lead to elliptical polarization because of the different velocities of propagation of the combined waves (*). In this model, the initial sinusoidal perturbation, for which the vector $\delta\mathbf{v}$ is arbitrarily directed in the plane x_2x_3, decompose into linearly polarized waves propagating in the direction $\pm x_1$. The components of the vector $\delta\mathbf{v}$ in the directions x_2 and x_3 propagate with different velocities.

For the model with the Jaumann-de Witt derivative, the relations between the amplitudes in fast waves are as follows:

$$\begin{gathered} w_1 = w_2 = 0, \qquad q = T_{13}w_{13}/c_+ \\ \sigma_{11} = T_{13}w_3/c_+, \quad \sigma_{13} = -\rho c_+ w_3, \quad \sigma_{23} = -{}^1/_2 T_{12}w_3/c_+, \quad \sigma_{33} = -\sigma_{11} \\ \sigma_{12} = \sigma_{22} = 0, \quad c_+ = \rho^{-1/2}\,[{}^1/_2(T_{11} - T_{33}) + \eta b/a]^{1/2} \end{gathered} \tag{3.4}$$

For slow waves in the same model, we obtain

$$\begin{gathered} w_1 = w_3 = 0, \qquad q = T_{12}w_2/c_- \\ \sigma_{11} = T_{12}w_2/c_-, \quad \sigma_{12} = -\rho c_- w_2, \quad \sigma_{22} = -\sigma_{11}, \quad \sigma_{23} = -{}^1/_2 T_{13}w_2/c_- \end{gathered} \tag{3.5}$$

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*) Exception from this rule is the case when the direction of the x_1-axis is such that $T_{22} = T_{33}$, i.e. $c_- = c_+$. For an arbitrary tensor $\mathbf{T}$, when $T_1 > T_2 > T_3$, there are always two such directions. (In crystal optics, similar directions define the principal axes of a biaxial crystal).

$$\sigma_{13} = \sigma_{33} = 0, \qquad c_- = \pm \rho^{-1/2} [{}^1/_2(T_{11} - T_{22}) + \eta b/a]^{1/2} \quad \text{(cont.)}$$

The implications of relations (3.4) and (3.5) concerning polarization of the waves is completely analogous with those which hold for the "covariant" model.

We note that in waves corresponding to the "Jaumann" model, the tensor $\delta\mathbf{T}$ is a deviator (*).

We proceed to the problem of propagation of three-dimensional short-wave perturbations. Let the initial condition for perturbation of a certain hydrodynamic function represent a wave packet combined from short-wave harmonics, which have identical wavelength and which depend on all kinds of directions. The initial distribution can be represented in the form

$$U(\mathbf{x}, 0) = \mathrm{Re}\left\{\iint\limits_{|\mathbf{n}|=1} \exp(ik\mathbf{n}\mathbf{x}) L(\varphi, \vartheta) \sin\vartheta \, d\varphi \, d\vartheta\right\} \tag{3.6}$$

Here the integral is taken with respect to unit sphere $|\mathbf{n}| = 1$; $L(\varphi, \vartheta)$ is a spectral function. Expression (3.6) follows from expansion of a perturbation of given type into a three-dimensional Fourier integral. The development of initial perturbation (3.6) with time will take place in accordance with the formula

$$U(\mathbf{x}, t) = \mathrm{Re}\left\{\iint\limits_{|\mathbf{n}|=1} \exp(ik[\mathbf{n}\mathbf{x} - (v_n + c(\mathbf{n}))t]) L(\varphi, \vartheta) \sin\vartheta \, d\varphi \, d\vartheta\right\}$$

Anisotropy of sound will have some effect on the geometry of the wave front in the propagation of such a perturbation.

We shall introduce the Cartesian system of coordinates with origin at the point $\mathbf{x}_0$ being examined, moving with an unperturbed velocity $\mathbf{v}(\mathbf{x}_0)$. The coordinate axes of this system R are directed along the principal axes of the unperturbed tensor $\mathbf{T}$. Let $\mathbf{r}$ be the radius vector of an arbitrary point relative to the origin of the system R. The surface defined by the vector equation $\mathbf{r} = c(\mathbf{n})\mathbf{n}$, where the parameter $\mathbf{n}$ passes the unit sphere, represents the phase polar. In the "contravariant" model the phase polar is obtained by transformation of the inversion relative to the sphere with center at the point $\mathbf{x}_0$ of an ellipsoid with the same center, for which the system R is found to be a system of the principal axes.

Let us suppose that the initial distribution $U(\mathbf{x}, 0)$ has a singularity at the point $\mathbf{x} = \mathbf{x}_0$. At the instant t, the "contribution" of this singularity in an elementary plane wave, the front of which is propagating in the direction $\mathbf{n}$, is concentrated on the planes orthogonal to the vector $\mathbf{n}$ and passing through the extremity of the vector $\mathbf{x}_0 + (\mathbf{v} + c\mathbf{n})t$. These planes, constructed for all possible directions $\mathbf{n}$, form a two-parameter set. The envelope of this set is the surface of the wave front which carries on it the singularity.

In the "contravariant" model the wave front is an ellipsoid, for which the system R coincides with the system of the principal axes. The magnitudes of the semi-axes of this ellipsoid at the instant $t = 1$ are equal to the velocities of sound in the principal directions of the tensor $\mathbf{T}$. The wave front at $t = 1$ is a group polar, formed by the

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*) In the case when we use the Williams-Bird operator [5] in model (1.3) instead of the operator (1.5), the tensor $\delta\mathrm{T}$ is also a deviator, and the condition for evolutionarity of model (1.3) for the case $r = s + 1$ has the form of (1.13).

extremities of the group velocity vectors $\mathbf{V} = \partial\omega/\partial\mathbf{k}$. The relative group velocity in the R system of coordinates is

$$\mathbf{V} = \left[\rho\left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2 + \eta\frac{b}{a}\right)\right]^{-1/2} \sum_{i=1}^{3}\left(T_i + \eta\frac{b}{a}\right) n_i \mathbf{e}_i$$

Here the unit vectors $\mathbf{e}_i$ form the base of the system R ; n_i are the components of the vector $\mathbf{n}$ in this base. The group and phase velocities coincide only for the principal directions of the tensor $\mathbf{T}$, and the group polar is contained within the phase polar.

For the "covariant" and "Jaumann" models, there exist two types of phase and group polars: fast and slow. A similar division of polars also holds in magnetohydrodynamics [12]. The geometry of the wave fronts for models containing the differential operators (1.6) and (1.7) in the general case are more complex than in the "contravariant" model.

If the tensor $\mathbf{T}$ is plane, the vector $\delta\mathbf{v}$ lies in the same plane, and two-dimensional perturbations are propagated in it, then the geometry of the wave fronts is similar for all three models with the derivatives (1.5)–(1.7). In this case, one of the principal values of the tensor $\mathbf{T}$ is equal to zero. The wave fronts originating from the point, are ellipses for all three models and the principal axes are parallel to the principal direction of the tensor $\mathbf{T}$ lying in the stated plane. The wave front for $t = 1$ in the "covariant" and "Jaumann" models coincides with the slow group polar if $T_3 = 0$, and with the fast group polar if $T_1 = 0$. In the case when $T_2 = 0$, the wave front for $t = 1$ is found to be composed of fragments of fast and slow polars. Such a wave front structure in models with the derivatives (1.6) and (1.7) is due to the perpendicular polarization of the fast and slow waves.

We note that the orthogonality of the vectors $\delta\mathbf{v}_+$ and $\delta\mathbf{v}_-$ in fast and slow waves, the fronts of which are propagated in one direction, is characteristic also for other types of continua possessing anisotropy. This phenomenon occurs in the propagation of magnetosonic waves [13] and also in the propagation of elastic waves in crystals [14].

In the light of what has been said, we shall consider the problem concerning the existence of characteristics in two-dimensional steady flow with a plane tensor $\mathbf{T}$ for the case of a two-constant model of a fluid. For definiteness, we shall consider the "contravariant" model (2.1). Examination of other models with a single time of relaxation is performed similarly.

Suppose the flow takes place in the plane xy, so that $T_{xz} = T_{yz} = T_{zz} = 0$ and $v_z = 0$, and all other flow parameters depend only on x and y. If we introduce the unknown vector-function $\mathbf{f} = (v_x,\ v_y,\ p,\ T_{xx},\ T_{xy},\ T_{yy})$, the entire system of hydrodynamic equations for the model (2.1) can be written in the form of the quasilinear differential equation

$$A(\mathbf{f})\frac{\partial\mathbf{f}}{\partial x} + B(\mathbf{f})\frac{\partial\mathbf{f}}{\partial y} + \mathbf{g}(\mathbf{f}) = 0$$

where A and B are square matrices of sixth order. If the characteristics of the system $y(x)$ satisfy the differential equation $dy/dx = \tau(x, y)$, then τ is a root of the characteristic equation

$$\det\|B - \tau A\| = 0$$

The roots of the latter equation have the form

$$\tau_{1,2} = v_y/v_x, \qquad \tau_{3,4} = \pm i$$

$$\tau_{5,6} = (-T_{xy}/\rho + v_x v_y \pm \sqrt{\xi})\,(v_x^2 - (T_{xx} + \eta/\lambda)/\rho)^{-1}$$

$$\xi = c_1^2 c_2^2 (v_1^2/c_1^2 + v_2^2/c_2^2 - 1) \tag{3.7}$$

Here v_1 and v_2 are components of the velocity vector in the system of the principal axes of the tensor $\mathbf{T}$; c_1 and c_2 are the magnitudes of the velocities of sound in the directions of the principal axes. Real characteristics which differ from streamlines exist only for the condition $\xi \geqslant 0$.

The following interpretation can be given to this fact. In steady flow, the nontrivial characteristics passing through the point $\mathbf{x}_0$ represent the wave fronts of perturbations originating from moving particle coinciding with the point $\mathbf{x}_0$ for $t = 0$. These stationary wave fronts are the envelopes of a set of elementaty wave fronts, representing ellipses with centers at the points $\mathbf{x}_0 + \mathbf{v}\,(\mathbf{x}_0)t$. All these ellipses are similar, and their principal axes are parallel to the principal directions of the tensor $\mathbf{T}\,(\mathbf{x}_0)$. The elementary wave fronts expand with time and for $t = 1$, such a wave front coincides with the group polar.

It is clear that a given set of ellipses has an envelope only in the case when from the point $\mathbf{x}_0$, a tangent to the group polar can be drawn with center at the point $\mathbf{x}_0 + \mathbf{v}\,(\mathbf{x}_0)$, i.e. when the point $\mathbf{x}_0$ is not located inside the group polar. The latter means that at the point $\mathbf{x}_0$ the quantity ξ is nonnegative and the characteristic roots $\tau_{5,6}$ determined by the formulas (3.7) are real.

The situation considered is similar to that which occurs in conventional gas dynamics, where characteristics which are different from the streamlines of steady two-dimensional flow exist only in supersonic regions.

We note that the system of equations for plane flow for model (2.1) with the plane tensor $\mathbf{T}$ does not become hyperbolic even in the case of supersonic flow, because of the presence of the imaginary characteristic roots $\tau_{3,4}$.

4. Reflection and refraction of high-frequency transverse wave of a viscoelastic fluid incident on an elastic wall. It is not difficult to foresee that the phenomenon of sound anisotropy in a viscoelastic fluid, possessing transient elasticity, should influence the nature of the reflection and refraction of waves at the boundary of such a fluid with other media.

Suppose, for example, that an incompressible fluid, corresponding to the "contravariant" model (2.1) and a classical linearly elastic solid are separated by a plane boundary G. We shall assume for the positive direction of the normal $\mathbf{v}$ to the interface of the media, the direction from the fluid to the solid body.

We shall denote by a dash the quantities referring to the elastic medium and we shall assume the classical equation of state

$$P_{ij}' = \varkappa' \operatorname{div} \mathbf{u}' \delta_{ij} + 2\mu' e_{ij}' \qquad (4.1)$$

for the elastic solid.

The relations (4.1) are written in Cartesian coordinates. Here P_{ij}' are components of the stress tensor, $\mathbf{u}'$ is the displacement vector, $\varkappa'$ and μ' are the Lamé constants, e_{ij}' are the components of the tensor of the infinitely small deformations.

Suppose there is a certain combined movement of the media (2.1) and (4.1), for which the interface G is the contact surface. Then the parameters of this movement should satisfy the laws of conservation of mass, momentum and energy at points of the surface G

$$v_\nu = v_\nu' = 0, \quad \mathbf{p}_\nu = \mathbf{p}_\nu', \quad \mathbf{p}_\nu \mathbf{v} = \mathbf{p}_\nu' \mathbf{v}' \qquad (4.2)$$

The latter relation in (4.2) has been written with the omission of the thermal conductivity of the media. In (4.2), $\mathbf{p}_\nu$ and $\mathbf{p}_\nu'$ are the vectors of the stress at the contact surface for each medium.

We shall assume that short-wave perturbations are excited in medium (2.1). As these perturbations are propagated with finite velocity, they are high-frequency perturbations. Since the decrement of the oscillations in an Oldroyd fluid (2.1) tends to zero as $k \to \infty$, then for sufficiently short waves we can neglect damping over a given period of time. Therefore, small perturbations of the parameters of both media can be assumed identical sinusoidal functions of time at points on the interface.

Waves, propagating in the fluid, on reaching the interface will be reflected from it and will excite waves in the elastic body. We emphasize that all future results will be of a localized nature and their range of application will be restriced to a quite small vicinity of an arbitrary point of the media interface.

In the vicinity of the arbitrary point $\mathbf{x}_0 \in G$, we shall seek solutions for the perturbations in the form (*)

$$(\delta v_i, \delta p, \delta T_{ij}) = \mathrm{Re}\,[(w_i, q, \sigma_{ij})\, e^{i\,(\mathbf{kx}-\omega t)}], \qquad (\mathbf{x}-\mathbf{x}_0)\,\boldsymbol{\nu} < 0$$

$$(\delta v_i'', \delta p'', \delta T_{ij}'') = \mathrm{Re}\,[(w_i'', q'', \sigma_{ij}'')\, e^{i\,(\mathbf{k}''\mathbf{x}-\omega t)}], \qquad (\mathbf{x}-\mathbf{x}_0)\,\boldsymbol{\nu} < 0$$

$$(\delta v_i', \delta P_{ij}') = \mathrm{Re}\,[(w_i', \sigma_{ij}')\, e^{i\,(\mathbf{k}'\mathbf{x}-\omega t)}], \qquad (\mathbf{x}-\mathbf{x}_0)\,\boldsymbol{\nu} > 0$$

Here, the two primes denote the parameters of the reflected wave. We shall assume that the interface of the media remains to be the contact surface in the perturbed motion. Using the laws of conservation (4.2), we arrive at the following boundary conditions, which should satisfy the perturbations at points of the plane G

$$\begin{gathered} \delta v_\nu + \delta v_\nu'' = \delta v_\nu' = 0, \qquad \delta T_{\nu i} + \delta T_{\nu i}'' = \delta P_{\nu i}' \\ (\delta T_{\nu i} + \delta T_{\nu i}'')\,(\delta v_i + \delta v_i'') = \delta P_{\nu i}'\delta v_i' \end{gathered} \tag{4.3}$$

In (4.3) we use the circumstance that the pressure perturbation in the wave corresponding to the "contravariant" model is equal to zero. It follows from (4.3) that at the points $\mathbf{x} \in G$ the eikonals of the incident, reflected and refracted waves, should coincide, i.e.

$$\mathbf{kx} = \mathbf{k}''\mathbf{x} = \mathbf{k}'\mathbf{x}$$

The latter condition means that the vectors $\mathbf{k}$, $\mathbf{k}''$, $\mathbf{k}'$ and $\boldsymbol{\nu}$ lie in the same plane and are reduced to the form

$$k \sin\psi = k''\sin\psi'' = k'\sin\psi' \tag{4.4}$$

Here ψ and ψ' are the angles between the vectors $\mathbf{k}$, $\boldsymbol{\nu}$ and $\mathbf{k}'$, $\boldsymbol{\nu}$, respectively; ψ'' is the angle between the vectors $\mathbf{k}''$, $-\boldsymbol{\nu}$.

The formulas

$$c = \left[\frac{1}{\rho}\,(T_{nn} + \mu)\right]^{1/2}, \qquad c'' = \left[\frac{1}{\rho}\,(T_{n''n''} + \mu)\right]^{1/2}, \qquad c' = \left(\frac{\mu'}{\rho'}\right)^{1/2}$$

are valid for the phase velocities of propagation of the waves.

Here T_{nn} and $T_{n''n''}$ are the normal "viscoelastic" stresses of unperturbed flow on areas which are orthogonal with the unit vectors $\mathbf{n} = \mathbf{k}/k$ and $\mathbf{n}'' = \mathbf{k}''/k$ respectively; $\mu = \eta/\lambda$. The quantity c' is equal to the velocity of the shear elastic waves.

In the case of no-slip condition of the fluid at the stationary elastic wall taking place in unperturbed flow, we obtain the laws of reflection and refraction in the form

$$\sin\psi/\sin\psi'' = c/c'', \qquad \sin\psi/\sin\psi' = c/c' \tag{4.5}$$

We introduce the auxiliary Cartesian system of coordinates K with origin at the point

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*) The case considered here is when longitudinal refracted wave does not appear in the elastic body.

$\mathbf{x}_0 \in G$ being investigated. The coordinates of an arbitrary point in the system K are denoted by x_i^*. The axes x_1^* and x_2^* of the system K are orientated along pairs of mutually orthogonal unit vectors, which realize extrema of the quadratic form $T_{ij}n_in_j$ at the intersection of the plane of the vectors $\mathbf{k}$ and $\boldsymbol{\nu}$ with unit sphere $|\mathbf{n}| = 1$. We denote by T_1^* and T_2^* the normal "viscoelastic" stresses at the coordinate areas $x_2^*x_3^*$ and $x_1^*x_3^*$ respectively. Without restriction of generality, it can be assumed that $T_1^* \geqslant T_2^*$. We denote by χ the angle between the vector $\boldsymbol{\nu}$ and the positive direction of the axis x_1^*. Then the relations (4.5) can be reduced to the form

$$\frac{\sin\psi}{\sin\psi''} = \left(\frac{T_1^*\cos^2(\chi-\psi) + T_2^*\sin^2(\chi-\psi)+\mu}{T_1^*\cos^2(\chi+\psi) + T_2^*\sin^2(\chi+\psi)+\mu}\right)^{1/2}$$
$$\frac{\sin\psi}{\sin\psi'} = \left(\frac{\rho'}{\rho}\right)^{1/2}\left(\frac{T_1^*\cos^2(\chi-\psi) + T_2^*\sin^2(\chi-\psi)+\mu}{\mu'}\right)^{1/2} \tag{4.6}$$

It is easy to see that in the general case the first equation of (4.6) is not satisfied for $\psi'' = \psi$. This effect is due to the anisotropy of sound in the Oldroyd medium (2.1). In a linearly Maxwellian fluid $c'' = c$ and $\psi'' = \psi$. It is obvious that the form of the law of reflection is independent of the properties of the second medium by which the fluid is bounded.

In order to determine the angle ψ'', we arrive at a quadratic equation with respect to $z = \operatorname{ctg}\psi''$, the roots of which are

$$z_1 = -\operatorname{ctg}\psi, \qquad z_2 = \operatorname{ctg}\psi + \frac{(T_1^* - T_2^*)\sin 2\chi}{T_1^*\cos^2\chi + T_2^*\sin^2\chi + \mu}$$

The first root z_1 must be discarded, as it leads to the initial incident wave. The second root z_2 determines the direction of propagation of the reflected wave front. The positive direction x_1^* can be chosen such that the angle χ is varied within the limits $0 \leqslant \chi < \pi$. Then, for $\chi = 0$ and $\chi = {}^1\!/_2\pi$, the equation $\psi'' = \psi$ is satisfied, i.e. the classical law of reflection holds for the normals to the wave fronts. For $0 < \chi < {}^1\!/_2\pi$, the inequality $\psi'' < \psi$ is satisfied while for ${}^1\!/_2\pi < \chi < \pi$ the inequality $\psi'' > \psi$ is satisfied.

In an anisotropic medium the wave excitation formed from harmonics whose wave vectors have infinitely close directions, will propagate with group velocity $\mathbf{V}(\mathbf{n})$. The unit vector $\mathbf{l} = \mathbf{V}/V$ will be called, as usual, the direction vector of the ray.

In the R system of the principal axes of the tensor $\mathbf{T}$, unit vectors of the normal to the plane wave front and the corresponding ray have the form

$$\mathbf{n} = \sum_{i=1}^{3} n_i\mathbf{e}_i, \qquad \mathbf{l} = \left(\sum_{i=1}^{3} c_i^2 n_i\mathbf{e}_i\right)\left(\sum_{i=1}^{3} c_i^4 n_i^2\right)^{-1/2}$$

Here c_i^2 are the squares of the velocity of sound in the principal directions. The vectors $\mathbf{n}$, $\mathbf{l}$ and $\boldsymbol{\nu}$ are not coplanar in the general case. These three vectors are coplanar only for the condition

$$(c_1^2 - c_2^2)\nu_3 n_1 n_2 + (c_2^2 - c_3^2)\nu_1 n_2 n_3 + (c_3^2 - c_1^2)\nu_2 n_3 n_1 = 0 \tag{4.7}$$

In the general case, Eq. (4.7) defines the cone of the directions $\mathbf{n}$. In the case when the vector $\boldsymbol{\nu}$ lies in one of the principal planes of the tensor $\mathbf{T}$, and likewise in the case of coincidence of the two principal velocities of sound, Eq. (4.7) defines two mutually orthogonal planes of the directions $\mathbf{n}$ intersecting along the vector $\boldsymbol{\nu}$; one of the planes is a principal plane of the tensor $\mathbf{T}$.

The directions of the unit vectors of the normal to the front of the reflected wave and the reflected ray are determined by the formulas

$$\mathbf{n}'' = \frac{\sin\psi''}{\sin\psi}\mathbf{n} - \frac{\sin(\psi+\psi'')}{\sin\psi}\mathbf{v}, \qquad \mathbf{l}'' = \left(\sum_{i=1}^{3} c_i^2 n_i''\mathbf{e}_i\right)\left(\sum_{i=1}^{3} c_i^4 n_i''^2\right)^{-1/2}$$

In contrast from the vectors $\mathbf{n}$, $\mathbf{n}''$ and $\mathbf{v}$, which are always coplanar, the vectors $\mathbf{l}$, $\mathbf{l}''$ and $\mathbf{v}$ lie in the same plane only in the case when

$$c_1^2(c_2^2 - c_3^2)\nu_2\nu_3 n_1 + c_2^2(c_3^2 - c_1^2)\nu_3\nu_1 n_2 + c_3^2(c_1^2 - c_2^2)\nu_1\nu_2 n_3 = 0 \qquad (4.8)$$

The latter equation defines the plane of the directions $\mathbf{n}$. In the case when $\mathbf{v}$ lies in the principal plane, the plane (4.8) coincides with this principal plane. If the vector $\mathbf{v}$ is oriented with respect to one of the principal directions of the tensor $\mathbf{T}$, Eq. (4.8) is satisfied identically for any $\mathbf{n}$.

If the vectors $\mathbf{v}$ and $\mathbf{n}$ both lie in the same principal plane, all five vectors $\mathbf{v}$, $\mathbf{n}$, $\mathbf{n}''$ $\mathbf{l}$ and $\mathbf{l}''$ will lie in this plane. If $\mathbf{v}$ does not lie in the principal plane, all the specified vectors are coplanar only for $\mathbf{n} = \mathbf{v}$.

As in the case of wave vectors, the calssical law of reflection is not valid for rays. In particular, a ray which is incident normally to the interface, can be reflected at an angle to the normal.

In what follows, we shall examine the case when in the unperturbed flow the surface G is at rest and there are no tangential stresses at this surface. In this case the direction of $\mathbf{v}$ will be the principal direction for the tensor $\mathbf{T}(\mathbf{x}_0)$, and $\psi = \psi''$. The angles of the incident and reflected rays with the normal to the interface will also be equal.

The coordinate axis x_1^* of the K system in this case can be assumed to be directed with respect to the vector $\mathbf{v}$. It is obvious that the inequality $T_1^* \geqslant T_2^*$ may or may not be fulfilled. It follows from the equation of state (2.1) and (4.1), and from relations (4.3) that $w_\nu = w_\nu'' = w_\nu' = 0$. From this and from the transversality condition of the waves it follows that velocity perturbations in the incident, reflected and refracted waves are colinear with the axis x_3^* of the K system. Therefore, all three waves are horizontal sheer waves.

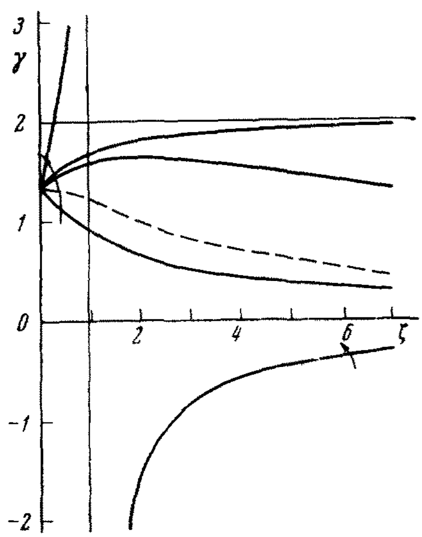

Fig. 1

We shall denote the amplitudes of the velocity perturbations in the incident, reflected and refracted waves by $w = w_3^*$, $w' = w_3'^*$ and $w'' = w_3''^*$ respectively. The relation

$$(w + w'' - w')\,[(w - w'')T_1^*\cos\psi + (w + w'')T_2^*\sin\psi] = 0 \qquad (4.9)$$

follows from the laws of conservation of momentum and energy.

If we equate the first cofactor in Eq. (4.9) to zero and use the law of conservation of momentum, and if we express the stress amplitudes in terms of the velocity amplitudes resulting from Eqs. (2.1), (4.1), we can determine the transparency coefficient $\gamma = w'/w$ and the coefficient of reflection $\varepsilon = w''/w = \gamma - 1$.

The expression for the transparency coefficient has the form

$$\gamma = \frac{2c_1^{*2}}{c_1^{*2} - (c_2^{*2} - c_0^2)\zeta + c'[c_1^{*2} + (c_2^{*2} - c'^2)\zeta^2]^{1/2}\rho'/\rho}, \qquad \zeta = \operatorname{tg}\psi \tag{4.10}$$

For sufficiently small relaxation times λ, the inequalities

$$c_1^* = [(T_1^* + \mu)/\rho]^{1/2} > c', \; c_2^* = [(T_2^* + \mu)/\rho]^{1/2} > c', \; c_0 = (\mu/\rho)^{1/2} = c' \tag{4.11}$$

can be assumed to be fulfilled.

The set of curves of $\gamma(\zeta)$ for different values of the parameter c_2^* for the case $c_1^* = c_0$ is plotted diagrammatically in Fig. 1. The arrows show the direction of increase of c_2^*. The dashed curve corresponds to a linear Maxwellian medium. When $c' < c_2^* < c_0$, the function $\gamma(\zeta)$ decreases monotonically with increase of ζ from 0 to ∞. In the case $c_0 < c_2^* < c_*$ the function $\gamma(\zeta)$ has a maximum at the point $\zeta = \zeta_*$. The following formulas are valid for the quantities c_2^* and ζ_*^2:

$$c_*^2 = c_0^2 + \frac{\rho'^2}{\rho^2}\frac{c'^2}{2} + \frac{\rho'}{\rho}c'(c_0^2 - {}^3/_4\, c'^2)^{1/2}$$

$$\zeta_*^2 = \frac{(c_2^{*2} - c_0^2)^2 c_1^{*2}}{(c_2^{*2} - c'^2)[c'^2(c_2^{*2} - c'^2)\rho'^2/\rho^2 - (c_2^{*2} - c_0^2)^2]}$$

When ζ_* varies from 0 to ∞, the values of $\gamma(\zeta_*)$ run through the interval $(\gamma(0), 2)$. For $c_2^* = c_*$ the function $\gamma(\zeta)$ is monotonic and $\lim \gamma(\zeta) = 2$ for $\zeta \to \infty$. In the case $c_* < c_2^* < \infty$ the distribution of $\gamma(\zeta)$ has singularity at the point $\zeta = \zeta_0$, where

$$\zeta_0 = \frac{c_1^*(c_1^{*2} - c'^2\rho'^2/\rho^2)}{c_1^*(c_2^{*2} - c_0^2) - c'[(c_2^{*2} - c_0^2)^2 + (c_2^{*2} - c'^2)(c_1^{*2} - c'^2\rho'^2/\rho^2)]^{1/2}\rho'/\rho}$$

When passing through the point ζ_0, the function $\gamma(\zeta)$ changes the sign ($+\infty$ for $-\infty$). For the angle $\psi_0 = \operatorname{arctg}\zeta_0$, the transparency and reflection coefficients become infinite. It can be assumed that for angles $\psi \approx \psi_0$ nonlinear effects become important, for the description of which linearization theory of small perturbations is unsuitable.

We note that by violation of the second inequality of (4.11), there is an angle of total internal reflection ψ_+ defined by the formula

$$\psi_+ = \arcsin[c_1^*(c'^2 - c_2^{*2} + c_1^{*2})^{-1/2}]$$

For $\psi > \psi_+$, absorption of the waves in the elastic body occurs, so that shear elastic waves propagate parallel to the interface of the media.

Vanishing of the second factor in (4.9) corresponds to the case $w' = 0$ and $\sigma'_{yi} = 0$. As the stress perturbations are equal to zero on the area which coincides with the interface, the stated case is also realized in the case of contact of a liquid with an ideal gas or vacuum. For the coefficient of reflection ε we obtain the formula

$$\varepsilon = \frac{c_1^{*2} - (c_2^{*2} - c_0^2)\zeta}{c_1^{*2} + (c_2^{*2} - c_0^2)\zeta}$$

In a linear Maxwellian liquid, the function $\varepsilon(\zeta) \equiv 1$. The effect of anisotropy of sound is expressed most sharply for oblique rays.

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